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# ON THE SIGNIFICANCE OF NORMAL CROSS-SECTIONAL EXTENSION IN BEAM THEORY WITH APPLICATION TO CONTACT PROBLEMS

P. M. NAGHDI
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, U.S.A.

and

M. B. RUBIN
Faculty of Mechanical Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel

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**Abstract**—The main features of contact problems of elastic beams are explored by considering a specific equilibrium problem of a beam in contact with a smooth rigid flat surface. Solutions of four separate linear theories, namely a general theory (G) and three others which are constructed as constrained theories are considered. These constrained theories differ from the general theory only by the degree of exclusion of one or both types of deformation usually referred to as (a) transverse normal strain and (b) transverse shear deformation. Thus, with both (a) and (b) absent, the constrained theory corresponds to the Bernoulli–Euler beam theory (BE); with only (b) absent, the constrained theory (N) accounts for normal extensional deformation; and with only (a) absent, the constrained theory corresponds to the Timoshenko beam theory (T). Comparison of the predictions of the solutions of the three constrained theories with that of theory G shows that theory N contains the main physics of the contact problem and correctly predicts the conditions under which the beam loses contact. Also the contact force is continuous at the end points of the contact region. In contrast, neither of the other two constrained theories (BE or T) correctly predicts these features.

## 1. INTRODUCTION

The content of this paper bears on the nature of contact problems of elastic beams, formulated here via a direct approach and in the context of a linearized equilibrium theory of beams with small deformation. In particular, for an isotropic, homogeneous, elastic beam of rectangular cross section under the action of applied bending moments at its end points, the evolution of contact region(s) on the lower surface of the beam in contact with a smooth rigid surface is explored as a result of an increase in the value of the applied moment.† This study is pursued by a direct approach in the theory of rods based on a *model*, known as a Cosserat (or directed) curve comprising a space curve and two directors representing the cross section of the rod. Although this model is three-dimensional in character, the basic equations which result from it depend only on one coordinate variable along the space curve. General background information on the direct formulation of the non-linear theories of rods can be found in Green *et al.* (1974b) and Naghdi (1982). In the present context, the two directors model the deformation of the material fibers (surrounding the space curve), which in the reference configuration of the rod are normal to the space curve and also orthogonal to each other. In the general non-linear theory of rods based on a Cosserat curve just described, the directors (or fibers) may experience three types of deformation as discussed recently (Naghdi and Rubin, 1984): (a) normal cross-sectional extension (or normal extension for brevity), (b) tangential shear deformation (or "transverse shear deformation"), and (c) normal cross-sectional shear deformation.

Often, some of the details provided by the general theory are not needed for the solution of a particular problem and it may be sufficient to use a simpler constrained theory which excludes one or more of deformations (a)–(c) mentioned above. A hierarchy of seven constrained theories is discussed in Naghdi and Rubin (1984). In the present paper, we are

† Additional shear forces must necessarily be applied to these end points in order to maintain their vertical positions when the beam is contacted.

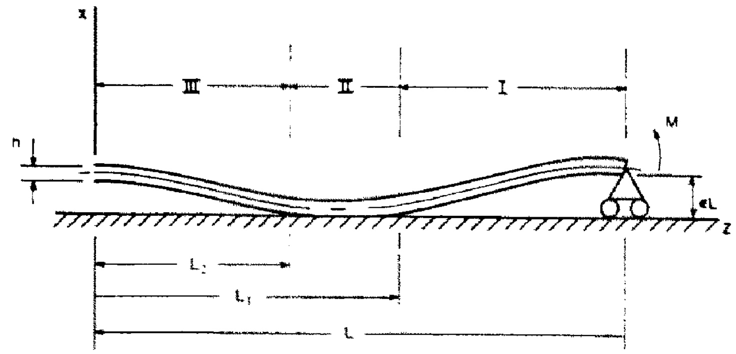


Fig. 1. A sketch of a beam under the action of a bending moment $M$ showing one half of the beam along $0 \leq z \leq L$ and showing contact region II and free regions I and III.

concerned with a straight, homogeneous, isotropic elastic beam of rectangular cross section undergoing planar motion without torsion. Consequently, we only need to consider the following three constrained theories: Constrained theory I, which excludes all three types of deformations (a)-(c) and corresponds to the Bernoulli–Euler beam theory; Constrained theory IV, which excludes deformations (b) and (c) but includes normal extension; and Constrained theory V, which excludes deformations (a) and (c) but includes the transverse shear deformation, and corresponds to the Timoshenko beam theory. For convenience, throughout the paper, we refer to the linearized version of the general (unconstrained) theory by G and to the linearized versions of each of the constrained theories I, IV, and V by BE, N, and T, respectively. For later reference, we include here the following.

*Statement of the problem.* Consider an elastic beam of rectangular cross section, parts of the lower surface of which may be in contact with a smooth stationary rigid flat surface. Let the beam be referred to a fixed system of rectangular Cartesian coordinates $x_i = (x, y, z)$ with associated orthonormal base vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$ and choose the origin of the coordinate system at the center of the beam with the $z$-axis directed to the right (Fig. 1). In its reference configuration, the beam is homogeneous and isotropic and is of length $2L$ in the $z$-direction, of height $h$ in the $x$-direction and width $w$ in the $y$-direction. The ends of the beam located at $(x, y, z) = (0, 0, \pm L)$: (i) are free to move in the $z$-direction, (ii) are fixed at an arbitrary height $x = \varepsilon L + h/2$—with $\varepsilon$ being a non-dimensional parameter much less than 1—above the rigid surface, (iii) are restrained from moving in the $y$-direction and (iv) are subjected to a bending moment $M$ in the $\pm y$-directions. The lateral surfaces of the beam are traction free except at points or regions of contact with the rigid surface. The effect of body force is neglected and for simplicity the center of the beam is regarded to be restrained from moving in the $z$-direction. The problem just described is symmetric about the $x$-axis in the $x$–$z$ plane as sketched in Fig. 1.

In the analysis that follows, we observe that as the moment $M$ increases from zero, the beam bends until $M$ equals a critical value $M^*$ when the beam first contacts the rigid surface at its center $z = 0$, $x = -h/2$. For values of $M$ larger than $M^*$ and less than a second critical value $M^{**}$ the contact region includes the center of the beam and extends over the region $z = \pm L_1$. For values of $M$ larger than $M^{**}$ the center of the beam loses contact and the contact region separates into the two regions $-L_1 < z < -L_2$ and $L_2 < z < L_1$ (Fig. 1). Considering only the positive half of the beam $(z > 0)$, we refer to these three cases as: no contact $(M < M^*)$; contact with one free region $(M^* \leq M \leq M^{**})$; and contact with two free regions $(M > M^{**})$. It may be noted that the analysis of separation of the contact region is relevant to the design of sockets for electrical devices because it can significantly influence the ability of a contact spring to maintain a good electrical connection.

Essenburg (1975) was the first to recognize the significance of including the effect of transverse normal strain in a similar contact problem. Starting from the three-dimensional equations and using an approximation procedure, Essenburg considered a slightly different problem of a beam with pinned ends and assumed a form for the stress distribution which satisfied boundary conditions pointwise on the lateral surfaces of the beam and in integrated form on the ends of the beam, and satisfied the equilibrium equations pointwise. He also assumed a displacement field which included quadratic dependence of the displacement on the thickness coordinate $x$ (in the notation of the present paper);† and, with the use of a variational procedure (in the three-dimensional theory), related the displacements (and their derivatives) to various stress resultants. The assumed form of the displacement field in Essenburg's paper includes the effects of both transverse normal extension and transverse shear deformation. In this regard, we may note that the general theory of a Cosserat surface (G) includes both the effects of normal extension and transverse shear deformation with displacements which have only a linear dependence on $x$. Consequently, in the context of the direct approach, the displacement field used by Essenburg (1975) corresponds to one associated with a more general direct theory (with more than two directors) than that used in the present paper.

In his paper, Essenburg (1975) showed that multiple contact regions can occur when the bending moment is increased in the manner described earlier (see the paragraph following the *Statement of the problem*). The validity of this result can be easily verified by a simple demonstration. To elaborate, consider two thin metal strips and separate them at each of their ends by bolts. Then, by using one's fingers to press the metal strips together on each of their ends just inside the bolts, it is possible to establish contact at the center of the strips and then cause separation there; this separation can be easily detected by observing that light passes between the strips. In the present paper, it is shown that for prediction of the loss of contact as described above it is necessary to include the normal extension effect and it is sufficient to use constrained theory N which isolates this effect. This result is intimately related to the Poisson effect and can be explained physically along the following lines. Thus, with reference to Fig. 1, we note that in a contact region the normal fibers (in the $x$-direction) near the bottom surface of the beam are compressed so axial fibers (in the $z$-direction) are extended due to the Poisson effect. Furthermore, since the top surface of the beam is traction free, the resultant axial load vanishes and the axial fibers there are not extended. It follows that this preferential extension of the axial fibers near the bottom surface of the beam (relative to its top surface) tends to create a curvature concave upward in the contact region. As the moment is increased the location of the maximum contact force per unit length in the positive half ($z > 0$) of the beam moves away from the beam's center ($z = 0$). Consequently, the tendency to create a concave upward curvature in the contact region is more significant near the location of the maximum contact force per unit length than near the beam's center, thereby causing the center of the beam to lose contact.

After some preliminary background information, the basic equations and constitutive equations of the general linearized theory G, along with the main features of the linear constrained theories BE, N and T, are summarized in Section 2. A detailed solution of the boundary-value problem described earlier in this section is obtained with the use of theory G in Section 3. The corresponding solutions with the use of the three constrained theories BE, N and T can be obtained similarly and are not recorded here. However, the results from all four solutions are discussed in Section 4 as illustrated also in Figs 2(a), (b) and 3. Basically, it is observed in Section 4 that the main physics of the contact problem are contained in theory N, which excludes the effect of transverse shear deformation but includes normal extension. Qualitatively, it predicts the same results as general theory G. In particular, it is found that the center of the beam loses contact for values of $M > M^{**}$ and the contact force between the beam and the rigid surface is continuous at the edges of the contact region, whereas theories BE and T do not correctly predict these features.

## 2. GENERAL BACKGROUND AND BASIC EQUATIONS

In this section we first provide brief background information on the non-linear theory of a Cosserat (or directed) curve with two directors and then summarize the main aspects of the linearized theory for a straight beam, along with linearized versions of the constrained theories mentioned in Section 1 (Naghdi and Rubin, 1984). A detailed discussion of the main kinematics and basic field equations of the theory of a Cosserat (or directed) curve

† The coordinates $(x, y, z)$ of the present paper correspond to $(z, -y, x)$ in Essenburg (1975).

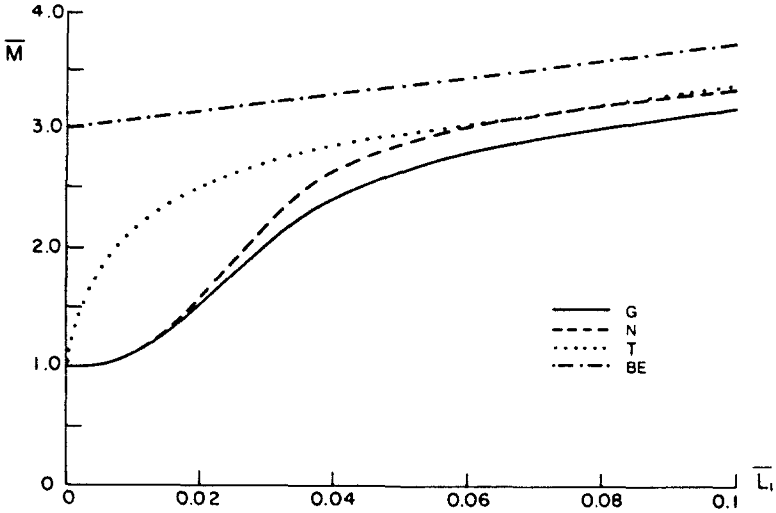


Fig. 2(a). Plots of the normalized location $\bar{L}_1 = L_1/L$ of the outer edge of the contact region as a function of the normalized moment $\bar{M}$ predicted by the general theory (G), by the theory (N) which accounts for normal extensional deformation, by the Timoshenko beam theory (T) and by the Bernoulli Euler beam theory (BE). Note that Fig. 2(a) represents an enlargement of a portion of Fig. 2(b).

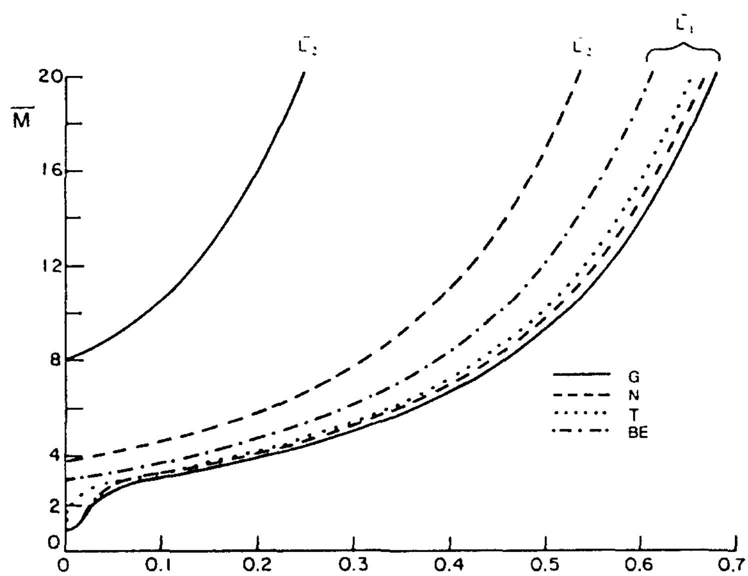


Fig. 2(b). Plots of the normalized locations $\bar{L}_1 = L_1/L$ of the outer edge of the contact region and $\bar{L}_2 = L_2/L$ of the inner edge of the contact region as functions of the normalized moment $\bar{M}$ predicted by all four theories (G, N, T and BE).

$\mathscr{R}$ which also models the deformation (and motion) of a linear elastic, homogeneous, isotropic straight beam may be found in Green *et al.* (1974a,b) and Naghdi (1982). There are some changes in the notations used in these papers and those of earlier ones on the subject. However, throughout this paper, we consistently use the notation of Naghdi (1982) and Naghdi and Rubin (1984) to which frequent references are made for conciseness.

Let the particles (material points) of the material curve $\mathscr{L}$ of $\mathscr{R}$ be identified by convected coordinate $\xi$; and, in the current configuration at time $t$, denote by $\mathbf{r}$ and $\mathbf{d}_\alpha$ ($\alpha = 1, 2$), respectively, the position vector of the material point and the directors at $\mathbf{r}$. The directors $\mathbf{d}_1$ and $\mathbf{d}_2$ represent the material fibers which in the reference configuration are parallel to the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ of the orthonormal basis $\mathbf{e}_i$ introduced in Section 1. For convenience, we introduce the notations

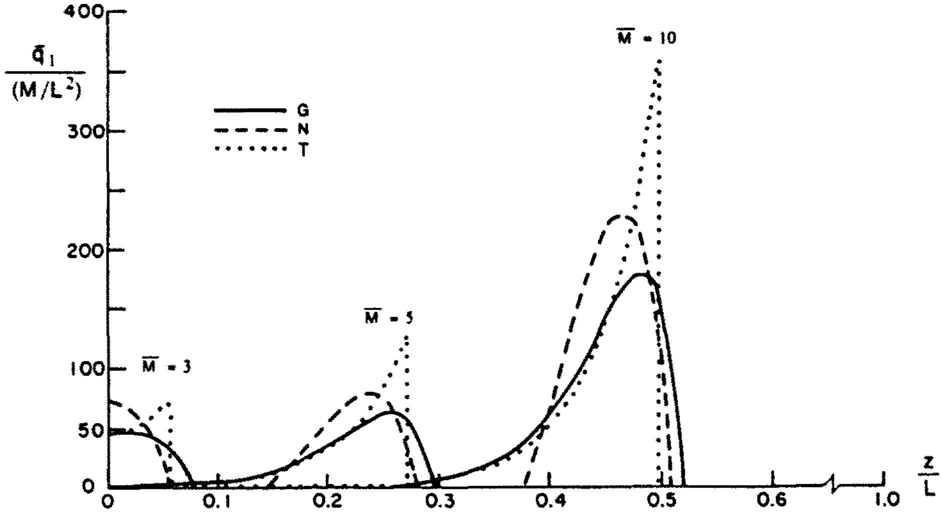


Fig. 3. Plots of $\bar{q}_1/(M/L^2)$ representing the normalized contact force $\bar{q}_1$ per unit length over the contact region for three different values of the normalized moment $\bar{M}$ as predicted by the three theories (G, N and T). The corresponding prediction by theory BE is not shown because the Bernoulli–Euler beam theory cannot predict this type of information.

$$\mathbf{d}_i = (\mathbf{d}_\alpha, \mathbf{d}_3), \qquad \mathbf{d}_3 = \frac{\partial \mathbf{r}}{\partial \xi} \tag{1}$$

and note that eqn $(1)_2$ represents the tangent vector to curve $\mathscr{R}$ in the current configuration. Further, the three vectors $\mathbf{d}_i$ are assumed to be linearly independent, i.e. $[\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3] > 0$.

Preparatory to the linearization of the various equations in the non-linear theory, let $\mathbf{u}$ and $\bar{\boldsymbol{\delta}}_i$ denote, respectively, the relative displacement of the material point $\xi$ and the relative director displacements at $\mathbf{r}$ defined by

$$\mathbf{r} = \mathbf{R} + \mathbf{u}, \qquad \mathbf{d}_i = \mathbf{D}_i + \bar{\boldsymbol{\delta}}_i \tag{2}$$

where†

$$\mathbf{u} = \mathbf{u}(\xi, t), \qquad \bar{\boldsymbol{\delta}}_i = \bar{\boldsymbol{\delta}}_i(\xi, t) \tag{3}$$

and where $\mathbf{R}$, $\mathbf{D}_i$ are the reference values of $\mathbf{r}$, $\mathbf{d}_i$ specified by

$$\mathbf{R} = \xi \mathbf{e}_3, \quad \mathbf{D}_i = \mathbf{e}_i. \tag{4}$$

All vector and tensor entities in this paper will be consistently referred to the base vectors $\mathbf{D}_i = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Thus, for example

$$\mathbf{u} = u_i \mathbf{e}_i, \qquad \bar{\boldsymbol{\delta}}_i = \bar{\delta}_{ij} \mathbf{e}_j \tag{5}$$

where the usual summation convention over repeated indices is employed. Using the notation of Naghdi and Rubin (1984), the linearized strain measures $\gamma_{ij}$, $\kappa_{\alpha i}$ are defined here by

† We use the symbol $\bar{\boldsymbol{\delta}}_i$ (rather than $\boldsymbol{\delta}_i$ without an overbar) to avoid possible confusion between the components $\bar{\delta}_{ij}$ of $\bar{\boldsymbol{\delta}}_i$ (see eqn $(5)_2$) with the usual notation for Kronecker delta representing the components of the unit tensor.

$$\gamma_{ij} = \frac{1}{2}(\delta_{ij} + \delta_{ji}), \qquad \kappa_{\alpha i} = \frac{\partial \delta_{\alpha i}}{\partial \xi}. \tag{6}$$

We also note that in the context of the constrained theories of rods (Naghdi and Rubin, 1984), the components† $n_i$ of the contact force $\mathbf{n}$ per unit length of the curve $\mathscr{L}$, the components $k_{\alpha i}$ of the instrinsic director force $\mathbf{k}_\alpha$ per unit length of $\mathscr{L}$, and the components $m_{\alpha i}$ of the contact force $\mathbf{m}_\alpha$ per unit length of $\mathscr{L}$, are determined to within additive constraint response such that

$$(\mathbf{n}, \mathbf{k}_\alpha, \mathbf{m}) = (\bar{\mathbf{n}}, \bar{\mathbf{k}}_\alpha, \bar{\mathbf{m}}_\alpha) + (\hat{\mathbf{n}}, \hat{\mathbf{k}}_\alpha, \hat{\mathbf{m}}_\alpha). \tag{7}$$

In eqn (7), the parts $(\hat{\mathbf{n}}, \hat{\mathbf{k}}_\alpha, \hat{\mathbf{m}}_\alpha)$ require constitutive equations and the other parts $(\bar{\mathbf{n}}, \bar{\mathbf{k}}_\alpha, \bar{\mathbf{m}}_\alpha)$ representing the constraint responses are workless and are determined by the equations of motion and the boundary conditions.

At this point, we may recall that the theory of a Cosserat curve can be brought into a one-to-one correspondence with that derived by approximation from the three-dimensional theory if the position vector $\mathbf{r}^*$ and the relative displacement $\mathbf{u}^*$ (in the three-dimensional theory) of an arbitrary point in the beam are linear functions of the convected coordinates $\theta^\alpha$ defining the cross section of the beam such that (Green *et al.*, 1974a)

$$\mathbf{r}^* = \mathbf{r}^*(\theta^\alpha, \xi, t) = \mathbf{r}(\xi, t) + \theta^\alpha \mathbf{d}_\alpha(\xi, t) \tag{8}$$

with

$$\mathbf{u}^* = \mathbf{u}^*(\theta^\alpha, \xi, t) = \mathbf{u}(\xi, t) + \theta^\alpha \boldsymbol{\delta}_\alpha(\xi, t). \tag{9}$$

The ranges of the coordinates $\theta^\alpha$ defining the rectangular cross section of the beam are

$$|\theta^1| \leqslant h/2, \qquad |\theta^2| \leqslant w/2 \tag{10}$$

where $\theta^1 = h/2$ defines the top surface and $\theta^1 = -h/2$ defines the bottom surface of the beam (Fig. 1). It follows from eqns (9) and (10) that the displacements $\hat{\mathbf{u}}$ and $\bar{\mathbf{u}}$ on the centerlines of the top and bottom surfaces, respectively, are given by

$$\hat{\mathbf{u}} = \mathbf{u} + \left(\frac{h}{2}\right)\boldsymbol{\delta}_1, \qquad \bar{\mathbf{u}} = \mathbf{u} - \left(\frac{h}{2}\right)\boldsymbol{\delta}_1. \tag{11}$$

As in the paper of Green *et al.* (1974a), the components of $(\mathbf{n}, \mathbf{k}_\alpha, \mathbf{m}_\alpha)$ may be identified with definitions of corresponding resultants which occur in the derivation of equations of equilibrium (or motion) from the three-dimensional equations; and, similarly, most of the constitutive coefficients in the direct approach may also be identified with the help of results obtained from the three-dimensional theory (Green *et al.*, 1974a,b; Naghdi, 1982).

In what follows, we directly quote from the results for the linearized theory of straight beams of isotropic materials from Green *et al.* (1974b), apart from some minor notational changes. In this connection, we note that (i) now $\lambda$ in Green *et al.* (1974a,b) and in Naghdi and Rubin (1984) is equal to the mass density per unit length of the beam, (ii) the strain measure $\gamma_{ij}$ in eqn (7.38) of Green *et al.* (1974b) is twice that defined by eqn $(6)_1$ of the present paper, and (iii) the constant constitutive coefficients $k_1, \ldots, k_{17}$ in Green *et al.* (1974b) are denoted in the present paper by $\alpha_1, \ldots, \alpha_{17}$ to avoid possible confusion which may arise from the slight change in the definition of strains.

It should be clear from the *Statement of the problem* in Section 1 (see also Fig. 1) that the type of deformation under consideration and characterized by eqns (5) must ensure

† Since in the linear theory all tensor quantities may be referred to the constant orthonormal base vectors $\mathbf{D}_i = \mathbf{e}_i$, there is no need to distinguish between covariant and contravariant components.

that: (i) the centerline $\mathscr{L}$ must always remain in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane, (ii) all cross-sectional fibers which were originally in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane must remain in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane, and (iii) cross-sectional fibers originally parallel to the $\mathbf{e}_2$-direction remain parallel to the $\mathbf{e}_2$-direction but are allowed to extend. In order to reflect these properties, we impose the following restrictions on the relative displacement $\mathbf{u}$ and relative director displacements $\boldsymbol{\delta}_\alpha$ :

$$\mathbf{u}\cdot\mathbf{e}_2 = 0, \qquad \boldsymbol{\delta}_1\cdot\mathbf{e}_2 = 0 \tag{12a,b}$$

$$\boldsymbol{\delta}_2\cdot\mathbf{e}_1 = 0, \qquad \boldsymbol{\delta}_2\cdot\mathbf{e}_3 = 0. \tag{12c,d}$$

Demanded by the symmetry of the problem defined in Section 1, the relevant constitutive equations for the isotropic beam under discussion are

$$\hat{n}_3 = \left(\alpha_8\delta_{11} + \alpha_9\delta_{22} + \frac{\alpha_3\,\partial u_3}{\partial z}\right) \tag{13a}$$

$$\hat{k}_{11} = \left(\alpha_1\delta_{11} + \alpha_7\delta_{22} + \frac{\alpha_8\,\partial u_3}{\partial z}\right) \tag{13b}$$

$$\hat{k}_{22} = \left(\alpha_7\delta_{11} + \alpha_2\delta_{22} + \frac{\alpha_9\,\partial u_3}{\partial z}\right) \tag{13c}$$

$$\hat{k}_{13} = \alpha_6\left(\delta_{13} + \frac{\partial u_1}{\partial z}\right) \tag{13d}$$

$$\hat{m}_{11} = \frac{\alpha_{10}\,\partial\delta_{11}}{\partial z} + \frac{\alpha_{17}\,\partial\delta_{22}}{\partial z} \tag{13e}$$

$$\hat{m}_{22} = \frac{\alpha_{17}\,\partial\delta_{11}}{\partial z} + \frac{\alpha_{11}\,\partial\delta_{22}}{\partial z} \tag{13f}$$

$$\hat{m}_{13} = \frac{\alpha_{16}\,\partial\delta_{13}}{\partial z} \tag{13g}$$

where in eqns (13) and in the remainder of the paper we have replaced the variable $\xi$ by $z$ of rectangular Cartesian coordinates introduced in Section 1.

Recalling from eqns (2.22) of Green *et al.* (1974a) that the assigned fields $\mathbf{f}$ and $\mathbf{l}^\alpha$ include contributions of both the three-dimensional body force and surface tractions on the lateral surfaces of the beam, then using eqns (12) and (13) and in the absence of the effect of the body force, the relevant equations of equilibrium (see eqns (2.7)–(2.9) of Naghdi and Rubin (1984)) for our present purpose may be displayed as

$$\frac{\partial n_1}{\partial z} + \hat{q}_1 + \bar{q}_1 = 0 \tag{14a}$$

$$\frac{\partial n_3}{\partial z} + \hat{q}_3 + \bar{q}_3 = 0 \tag{14b}$$

$$\frac{\partial m_{11}}{\partial z} + \left(\frac{h}{2}\right)(\hat{q}_1 - \bar{q}_1) - \hat{k}_{11} - \bar{k}_{11} = 0 \tag{14c}$$

$$\frac{\partial m_{22}}{\partial z} - \hat{k}_{22} - \bar{k}_{22} = 0 \tag{14d}$$

$$\frac{\partial m_{13}}{\partial z} + \left(\frac{h}{2}\right)(\hat{q}_3 - \bar{q}_3) - \hat{k}_{13} - \bar{k}_{13} = 0 \tag{14e}$$

$$n_1 = \frac{\partial m_{13}}{\partial z} + \left(\frac{h}{2}\right)(\hat{q}_3 - \bar{q}_3). \tag{14f}$$

In the above equations of equilibrium for beams, eqns (14a) and (14b) are consequences of linear momentum, eqns (14c)–(14e) are consequences of director momentum and eqn (14f) arises from the moment of momentum. Also, $\hat{q}_1$ and $\hat{q}_3$ are components of the applied force per unit length at the top surface ($x = h/2$) of the beam, while $\bar{q}_1$ and $\bar{q}_3$ are the components of the applied force per unit length at the bottom surface ($x = -h/2$) of the beam. Depending on whether or not constraints are imposed on the directors we can discuss four types of theories: we refer to these as the general theory (G), the Bernoulli–Euler theory (BE), the normal extensional theory (N), and the Timoshenko beam theory (T). A feature which is common to all four theories is that the component $n_1$ is determined by eqn (14f) and does not require a constitutive equation.† Moreover, the constraint responses $\bar{n}_3$ and $\bar{m}_{\alpha i}$ vanish (for details see Naghdi and Rubin (1984)) so that

$$n_3 = \hat{n}_3, \qquad m_{\alpha i} = \hat{m}_{\alpha i}. \tag{15}$$

Before proceeding further, it is desirable to indicate briefly the structure of these theories which may be described as follows.

*General theory* (G). For the general theory, there are no constraints and in the absence of constraint responses, the deformation fields in terms of the components $(u_1, u_3, \delta_{11}, \delta_{13}, \delta_{22})$ are determined by all five differential equations, eqns (14a)–(14e) with $\bar{k}_{11} = \bar{k}_{22} = \bar{k}_{13} = 0$, while eqn (14f) determines $n_1$ (as was noted above).

*Constrained theory* (BE). This theory excludes both normal extensional and transverse shear deformation by imposing the kinematical constraints

$$\gamma_{11} = 0, \qquad \gamma_{22} = 0, \qquad \gamma_{13} = 0. \tag{16}$$

The relevant system of governing equations in this case simplifies considerably: the deformation fields $(u_1, u_3)$ are determined by the two differential equations, eqns (14a) and (14b), while the remaining three equations, eqns (14c)–(14e) determine the constraint responses $\bar{k}_{11}, \bar{k}_{22}, \bar{k}_{13}$.

*Constrained theory* (N). This theory includes normal extensional deformation, but excludes transverse shear deformation by imposing only constraint $(16)_3$. The deformation fields $(u_1, u_3, \delta_{11}, \delta_{22})$ in this case are determined by the four differential equations, eqns (14a)–(14d) with $\bar{k}_{11} = \bar{k}_{22} = 0$, while eqn (14e) may be regarded as an equation for the determination of the constraint response $\bar{k}_{13}$.

*Constrained theory* (T). This theory includes the effect of transverse shear deformation but excludes normal extension by imposing constraints $(16)_{1,2}$. The deformation fields $(u_1, u_3, \delta_{13})$ in this case are determined by the three differential equations, eqns (14a), (14b) and (14e) with $\bar{k}_{13} = 0$, while eqns (14c) and (14d) determine the constraint responses $\bar{k}_{11}$ and $\bar{k}_{22}$.

Most of the constitutive coefficients which occur in the direct formulation of the theory of elastic rods have already been identified in the paper of Green *et al.* (1974b). Apart from some notational changes and a slight change in the definition for $\gamma_{ij}$ mentioned earlier, these coefficients are given by

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{Ehw(1-\nu)}{(1+\nu)(1-2\nu)} \tag{17}$$

$$\alpha_7 = \alpha_8 = \alpha_9 = \frac{\nu\alpha_1}{(1-\nu)} \tag{18}$$

† This is consistent with a general procedure in continuum mechanics according to which moment of momentum equation is regarded to be satisfied identically by all field quantities in the theory.

and

$$\alpha_{16} = \frac{Eh^3 w}{12}, \qquad \alpha_6 = \frac{5}{6}\mu h w \tag{19}$$

where $E$ is Young's modulus of elasticity, $v$ is Poisson's ratio and $\mu = E/2(1+v)$ is the shear modulus. The coefficients in eqns (17) and (18) were identified in Green *et al.* (1974b) by comparison with exact solutions in the three-dimensional theory for simple extensional deformation of a rod in the $\mathbf{e}_1$- and $\mathbf{e}_3$-directions, while eqn $(19)_1$ was identified by comparison with the exact solution for pure bending of a beam. The coefficient $\alpha_6$ is associated with the effect of transverse shear and its approximate value, as recorded in eqn $(19)_2$, is commonly used in the literature for static solutions which include the effect of transverse shear deformation. The remaining coefficients in the constitutive equations, eqns (13), namely $\alpha_{10}, \alpha_{11}, \alpha_{17}$ which occur in eqns (13e) and (13f), have not been identified so far by comparison from exact solutions in the three-dimensional theory. However, since the main physics of the contact problem is retained even in the absence of these coefficients, we set

$$\alpha_{10} = \alpha_{11} = \alpha_{17} = 0 \tag{20}$$

which also leads to a considerable simplification of the system of equations governing the contact problem under discussion. It is of interest to note that specification (20) is tantamount to a special constitutive assumption which renders the strain energy density function independent of kinematical quantities $\kappa_{11}$ and $\kappa_{22}$.

Specifications (17)–(20) are valid for constrained theory N which includes normal extensional deformation. However, for constrained theories BE and T which exclude normal extensional deformation we must specify $\alpha_3$ by

$$\alpha_3 = Ehw \tag{21}$$

in order to obtain the correct results for simple tension in the $\mathbf{e}_3$-direction. In this regard we may note that when the lateral surfaces of the beam are traction free, the flexural and extensional equations are decoupled. However, when the contact forces (e.g. $\hat{q}_1, \hat{q}_3$) are present, the flexural and extensional equations are coupled (see eqn (14a)). The value (21) is included for completeness even though it is not needed for the solutions presented here because the flexural and extensional equations of theories BE and T remain uncoupled.

Recalling the *Statement of the problem* in Section 1, for the contact under consideration, the top surface of the beam is free of contact force ($\hat{q}_1 = \hat{q}_3 = 0$), the contact surface is smooth ($\bar{q}_3 = 0$) and the ends $z = \pm L$ are allowed to extend freely in the $\mathbf{e}_3$-direction since $n_3 = 0$. With the help of eqns (13) and (20), the governing equations, eqns (14), reduce to

$$\bar{q}_1 = -\frac{\alpha_{16}\,\mathrm{d}^3\delta_{13}}{\mathrm{d}z^3} \tag{22a}$$

$$\alpha_8\delta_{11} + \alpha_9\delta_{22} + \frac{\alpha_3\,\mathrm{d}u_3}{\mathrm{d}z} = \text{constant} \tag{22b}$$

$$\bar{k}_{11} = \left(\frac{h}{2}\right)\frac{\alpha_{16}\,\mathrm{d}^3\delta_{13}}{\mathrm{d}z^3} - \left(\alpha_1\delta_{11} + \alpha_7\delta_{22} + \frac{\alpha_8\,\mathrm{d}u_3}{\mathrm{d}z}\right) \tag{22c}$$

$$\bar{k}_{22} = -\left(\alpha_7\delta_{11} + \alpha_2\delta_{22} + \frac{\alpha_9\,\mathrm{d}u_3}{\mathrm{d}z}\right) \tag{22d}$$

$$\bar{k}_{13} = \frac{\alpha_{16}\,\mathrm{d}^2\delta_{13}}{\mathrm{d}z^2} - \alpha_6\left(\delta_{13} + \frac{\mathrm{d}u_1}{\mathrm{d}z}\right) \tag{22e}$$

$$n_1 = \frac{\alpha_{16}\, d^2\bar{\delta}_{13}}{dz^2}. \tag{22f}$$

2.1. *Solution procedure*

The analysis of the contact problem with the use of general theory G naturally leads to the consideration of three separate regions: (i) the free region I, (ii) the contact region II, and (iii) the free region III (Fig. 1). Thus, we first consider the general solutions in each of these three regions and then impose appropriate boundary conditions at the end $z = L$, symmetry conditions at the center, and matching conditions at the boundaries $z = L_1$ and $L_2$.

Specifically, the boundary conditions at the end $z = L$ are

$$u_1(L) = 0, \qquad n_3(L) = 0, \qquad m_{13}(L) = -M \tag{23a–c}$$

and the symmetry conditions are

$$u_1(z) = u_1(-z), \qquad u_3(0) = 0, \qquad \bar{\delta}_{13}(z) = -\bar{\delta}_{13}(-z) \tag{24a–c}$$

$$\bar{\delta}_{11}(z) = \bar{\delta}_{11}(-z), \qquad \bar{\delta}_{22}(z) = \bar{\delta}_{22}(-z). \tag{24d,e}$$

Furthermore, the matching conditions at $z = L_1$ are

$$u_1(L_1^-) = u_1(L_1^+), \qquad u_3(L_1^-) = u_3(L_1^+), \qquad \bar{\delta}_{13}(L_1^-) = \bar{\delta}_{13}(L_1^+) \tag{25a-c}$$

$$\bar{\delta}_{11}(L_1^-) = \bar{\delta}_{11}(L_1^+), \qquad \bar{\delta}_{22}(L_1^-) = \bar{\delta}_{22}(L_1^+) \tag{25d,e}$$

$$n_1(L_1^-) = n_1(L_1^+), \qquad n_3(L_1^-) = n_3(L_1^+) \tag{25f,g}$$

$$m_{13}(L_1^-) = m_{13}(L_1^+), \qquad \bar{q}_1(L_1^-) = \bar{q}_1(L_1^+) = 0 \tag{25h,i}$$

$$\bar{u}_1(L_1^-) = \bar{u}_1(L_1^+) = -\varepsilon L \tag{25j}$$

and the matching conditions at $z = L_2$ are

$$u_1(L_2^-) = u_1(L_2^+), \qquad u_3(L_2^-) = u_3(L_2^+), \qquad \bar{\delta}_{13}(L_2^-) = \bar{\delta}_{13}(L_2^+) \tag{26a–c}$$

$$\bar{\delta}_{11}(L_2^-) = \bar{\delta}_{11}(L_2^+), \qquad \bar{\delta}_{22}(L_2^-) = \delta_{22}(L_2^+) \tag{26d,e}$$

$$n_1(L_2^-) = n_1(L_2^+), \qquad n_3(L_2^-) = n_3(L_2^+) \tag{26f,g}$$

$$m_{13}(L_2^-) = m_{13}(L_2^+), \qquad \bar{q}_1(L_2^-) = \bar{q}_1(L_2^+) = 0 \tag{26h,i}$$

$$\bar{u}_1(L_2^-) = \bar{u}_1(L_2^+) = -\varepsilon L. \tag{26j}$$

From the constitutive equations, eqns (13), conditions (15), the equations of equilibrium (22), as well as the conditions $\bar{k}_{\alpha\alpha} = 0$, it follows that the kinematical quantities

$$u_1, \quad \frac{du_1}{dz}, \quad u_3, \quad \frac{du_3}{dz}, \quad \bar{\delta}_{11}, \quad \bar{\delta}_{22} \tag{27a–f}$$

$$\bar{\delta}_{13}, \quad \frac{d\bar{\delta}_{13}}{dz}, \quad \frac{d^2\bar{\delta}_{13}}{dz^2}, \quad \frac{d^3\bar{\delta}_{13}}{dz^3} \tag{27g–j}$$

are continuous at the boundaries $z = L_1$ and $L_2$. Further, it follows from (11) and (27) that the kinematics of the top and bottom surfaces of the beam satisfy the conditions that†

† These conditions differ from those used by Essenburg (1975). His matching conditions are not sufficient to ensure the continuity of the displacement at the top surface of the beam.

$$\hat{u}_1, \quad \hat{u}_3, \quad \frac{d\hat{u}_3}{dz} \tag{28a–c}$$

$$\bar{u}_1, \quad \bar{u}_3, \quad \frac{d\bar{u}_3}{dz} \tag{28d–f}$$

are also continuous at the boundaries $z = L_1$ and $L_2$. It is of interest to note that the quantities $d\hat{u}_1/dz$ and $d\bar{u}_1/dz$ are not necessarily continuous because the quantities $d\bar{\delta}_{11}/dz$ and $d\bar{\delta}_{22}/dz$ are not necessarily continuous. However, if we were to consider a theory in which $\alpha_{10}$, $\alpha_{11}$, $\alpha_{17}$ were not assumed to vanish, then we would require continuity of $m_{11}$ and $m_{22}$ at the boundaries $z = L_1$ and $L_2$. This would then require continuity of $d\bar{\delta}_{11}/dz$, $d\bar{\delta}_{22}/dz$, $d\hat{u}_1/dz$ and $d\bar{u}_1/dz$, but this is not considered here.

It is fairly straightforward to show that constrained theory N satisfies the same continuity requirements as general theory G, whereas constrained theories T and BE do not. More specifically, $d^3\bar{\delta}_{13}/dz^3$ (or $\bar{q}_1$) is not continuous for constrained theory T and both $d^3\bar{\delta}_{13}/dz^3$ and $d^2\bar{\delta}_{13}/dz^2$ (or $\bar{q}_1$ and $\bar{n}_1$) are not continuous for constrained theory BE.

## 3. SOLUTION—GENERAL THEORY (G)

In this section we first obtain general solutions in the three regions, regions I–III, and then develop the solution of the contact problem for the three cases: no contact ($M < M^*$); contact with one free region ($M^* \leqslant M \leqslant M^{**}$); and contact with two free regions ($M > M^{**}$).

For the general theory there are no constraints and this is equivalent to setting

$$\bar{k}_{11} = \bar{k}_{22} = \bar{k}_{13} = 0. \tag{29}$$

Also, since the ends of the beam are free of axial force ($n_3 = 0$), and by eqns (25g) and (26g) the axial force must be continuous, we conclude that the constant in eqn (22b) vanishes so that

$$\frac{du_3}{dz} = -\left(\frac{\alpha_8}{\alpha_3}\right)\bar{\delta}_{11} - \left(\frac{\alpha_9}{\alpha_3}\right)\bar{\delta}_{22}. \tag{30}$$

Further, using eqns (29) and (30), eqns (22a) and (22c)–(22e) may be rewritten in the forms

$$\bar{\delta}_{22} = -\left(\frac{\alpha_3\alpha_7 - \alpha_8\alpha_9}{\alpha_2\alpha_3 - \alpha_9^2}\right)\bar{\delta}_{11} \tag{31a}$$

$$\frac{d^3\bar{\delta}_{13}}{dz^3} - \left(\frac{2\alpha}{h\alpha_{16}}\right)\bar{\delta}_{11} = 0 \tag{31b}$$

$$\bar{q}_1 = -\left(\frac{2\alpha}{h}\right)\bar{\delta}_{11} \tag{31c}$$

$$\frac{\alpha_{16}\, d^2\bar{\delta}_{13}}{dz^2} - \alpha_6\left(\bar{\delta}_{13} + \frac{du_1}{dz}\right) = 0 \tag{31d}$$

where the constant $\alpha$ in eqns (31b) and (31c) is defined by

$$\alpha = \frac{(\alpha_1\alpha_3 - \alpha_8^2)(\alpha_2\alpha_3 - \alpha_9^2) - (\alpha_3\alpha_7 - \alpha_8\alpha_9)^2}{\alpha_3(\alpha_2\alpha_3 - \alpha_9^2)}. \tag{32}$$

*Free region* I. For free region I the contact force $\bar{q}_1$ vanishes and the solution of eqns (30) and (31) subject to boundary conditions (23) and matching condition (25j) yields

$$u_1 = -\left(\frac{M}{2\alpha_{16}}\right)(L-z)(z-L_1) - \frac{\varepsilon L(L-z)}{(L-L_1)} - \left(\frac{a_1}{3}\right)(L-z)(z-L_1)(2L-L_1-z) \tag{33a}$$

$$u_3 = a_5, \qquad \bar{\delta}_{11} = 0, \qquad \bar{\delta}_{22} = 0 \tag{33b–d}$$

$$\bar{\delta}_{13} = -\frac{\varepsilon L}{(L-L_1)} + \left(\frac{M}{2\alpha_{16}}\right)(L+L_1-2z) + a_1\left[(z-L_1)(z+L_1-2L) + \frac{2(L-L_1)^2}{3} + \frac{2\alpha_{16}}{\alpha_6}\right] \tag{33e}$$

$$n_1 = 2\alpha_{16}a_1, \qquad n_3 = 0, \qquad \bar{q}_1 = 0 \tag{33f–h}$$

$$m_{13} = -M - 2a_1\alpha_{16}(L-z) \tag{33i}$$

where $a_1$ and $a_5$ are constants of integration.

*Contact region* II. For contact region II the normal displacement at the contact surface is (Fig. 1)

$$\bar{u}_1 = u_1 - \left(\frac{h}{2}\right)\bar{\delta}_{11} = -\varepsilon L. \tag{34}$$

Differentiating eqn (31b) with respect to $z$ and using eqns (31d) and (34) to eliminate $\bar{\delta}_{11}$ and $du_1/dz$ we deduce the equation

$$\frac{d^4\bar{\delta}_{13}}{dz^4} - 2b^2\frac{d^2\bar{\delta}_{13}}{dz^2} + a^4\bar{\delta}_{13} = 0 \tag{35}$$

where constants $a$ and $b$ are defined through

$$a^4 = \frac{4\alpha}{h^2\alpha_{16}}, \qquad b^2 = \frac{2\alpha}{h^2\alpha_6}. \tag{36a,b}$$

Using standard techniques a general solution of eqn (35) may be written in the form

$$\bar{\delta}_{13} = b_1 \cosh \beta_1(z-L_2) \cosh \beta_2(z-L_2) + b_2 \sinh \beta_1(z-L_2) \sin \beta_2(z-L_2) + b_3 \cosh \beta_1(z-L_2) \sin \beta_2(z-L_2) + b_4 \sinh \beta_1(z-L_2) \cos \beta_2(z-L_2) \tag{37}$$

where $b_1, b_2, b_3, b_4$ are constants of integration to be determined and constants $\beta_1$ and $\beta_2$ are given by

$$\beta_1 = a \cos \theta, \qquad \beta_2 = a \sin \theta \tag{38a,b}$$

$$\theta = \left(\frac{1}{2}\right)\tan^{-1}\left[\frac{(a^4-b^4)^{1/2}}{b^2}\right]. \tag{38c}$$

The remaining part of the solution for region II in terms of eqns (37) and (22) may be summarized as follows:

$$u_1 = -\varepsilon L + \left(\frac{h}{2}\right)\bar{\delta}_{11} \tag{39a}$$

$$u_3 = -\bar{\alpha}\left[\frac{d^2\bar{\delta}_{13}}{dz^2} - \frac{d^2\bar{\delta}_{13}(L_2)}{dz^2}\right] + b_5 \tag{39b}$$

$$\bar{\delta}_{11} = \left(\frac{h\alpha_{16}}{2\alpha}\right)\frac{d^3\bar{\delta}_{13}}{dz^3} \tag{39c}$$

$$\bar{\delta}_{22} = -\left(\frac{\alpha_3\alpha_7 - \alpha_8\alpha_9}{\alpha_2\alpha_3 - \alpha_9^2}\right)\bar{\delta}_{11} \tag{39d}$$

$$n_1 = \frac{\alpha_{16}}{} \frac{d^2\bar{\delta}_{13}}{dz^2}, \qquad n_3 = 0 \tag{39e,f}$$

$$\bar{q}_1 = -\left(\frac{2\alpha}{h}\right)\bar{\delta}_{11}, \qquad m_{13} = \alpha_{16}\frac{d\bar{\delta}_{13}}{dz} \tag{39g,h}$$

$$\bar{\alpha} = \left(\frac{h\alpha_{16}}{2\alpha}\right)\left[\frac{\alpha_8(\alpha_2\alpha_3 - \alpha_9^2) - \alpha_9(\alpha_3\alpha_7 - \alpha_8\alpha_9)}{\alpha_3(\alpha_2\alpha_3 - \alpha_9^2)}\right] \tag{39i}$$

where $\bar{\delta}_{13}$ is given by eqn (37) and $b_5$ in eqn (39b) is another constant of integration.

*Free region* III. For free region III the contact force $\bar{q}_1$ vanishes and a solution of eqns (30) and (31) subject to symmetry conditions (24) and matching condition (25j) yields

$$u_1 = (\tfrac{1}{2})c_2(L_2^2 - z^2) - \varepsilon L, \qquad u_3 = 0 \tag{40a,b}$$

$$\bar{\delta}_{11} = 0, \qquad \bar{\delta}_{22} = 0, \qquad \bar{\delta}_{13} = c_2 z \tag{40c–e}$$

$$n_1 = 0, \qquad n_3 = 0, \qquad \bar{q}_1 = 0 \tag{40f–h}$$

$$m_{13} = \alpha_{16}c_2 \tag{40i}$$

where $c_2$ is a constant of integration.

*Solution—no contact.* When there is no contact, the contact force $\bar{q}_1$ vanishes and a solution of eqns (30) and (31) subject to boundary conditions (23) and symmetry conditions (24) results in

$$u_1 = -\left(\frac{M}{2\alpha_{16}}\right)(L^2 - z^2), \qquad u_3 = 0 \tag{41a,b}$$

$$\bar{\delta}_{11} = 0, \qquad \bar{\delta}_{22} = 0, \qquad \bar{\delta}_{13} = -\left(\frac{M}{\alpha_{16}}\right)z \tag{41c–e}$$

$$n_1 = 0, \qquad n_3 = 0, \qquad \bar{q}_1 = 0 \tag{41f–h}$$

$$m_{13} = -M. \tag{41i}$$

We observe that contact will first occur at the center of the beam when $u_1(0) = -\varepsilon L$ and $M = M^*$ with $M^*$ given by

$$M^* = \frac{2\alpha_{16}\varepsilon}{L}. \tag{42}$$

Solution (41) corresponds to that for pure bending of a beam. With the help of kinematical expressions (6), (41a), and (41e), it can be readily demonstrated that the effect of transverse shear is entirely absent in this case, consistent with the known exact three-dimensional solution.

*Solution—contact with one free region.* The solution is characterized by eqns (33) in region I and eqns (37)–(39) in region II. Since there is only one free region we set

$$L_2 = 0 \tag{43}$$

and impose symmetry condition (24c) on the solution in region II to deduce that

$$b_1 = b_2 = 0. \tag{44}$$

The remaining constants are determined by symmetry condition (24b) and matching conditions (25b)–(25d), (25f) and (25h) and are given by

$$a_1 = A_1 b_3, \qquad a_5 = -\bar{\alpha}\left[\frac{d^2\bar{\delta}_{13}(L_1)}{dz^2} - \frac{d^2\bar{\delta}_{13}(0)}{dz^2}\right] \tag{45a,b}$$

$$b_3 = -B_3\left(\frac{M}{\alpha_{16}}\right), \qquad b_4 = B_4 b_3, \qquad b_5 = 0 \tag{45c–e}$$

$$\bar{M} = \left[\frac{L}{2(L-L_1)}\right]\left[\frac{(L-L_1)}{2L} + \left(\frac{B_3}{L}\right)\left(\cosh\beta_1 L_1 \sin\beta_2 L_1 + B_4 \sinh\beta_1 L_1 \cos\beta_2 L_1 - A_1\left\{\frac{2(L-L_1)^2}{3} + \frac{2\alpha_{16}}{\alpha_6}\right\}\right)\right]^{-1} \tag{45f}$$

where the normalized moment $\bar{M}$ and constants $A_1$, $B_3$, and $B_4$ are defined by

$$\bar{M} = \frac{M}{M^*} \tag{46a}$$

$$A_1 = (\tfrac{1}{2})[(\beta_1^2-\beta_2^2)-2B_4\beta_1\beta_2]\cosh\beta_1 L_1 \sin\beta_2 L_1 + (\tfrac{1}{2})[2\beta_1\beta_2 + B_4(\beta_1^2-\beta_2^2)]\sinh\beta_1 L_1 \cos\beta_2 L_1 \tag{46b}$$

$$B_3 = [2A_1(L-L_1) + (B_4\beta_1+\beta_2)\cosh\beta_1 L_1 \cos\beta_2 L_1 + (\beta_1 - B_4\beta_2)\sinh\beta_1 L_1 \sin\beta_2 L_1]^{-1} \tag{46c}$$

$$B_4 = \frac{(\beta_2^3-3\beta_1^2\beta_2)\cosh\beta_1 L_1 \cos\beta_2 L_1 - (\beta_1^3-3\beta_1\beta_2^2)\sinh\beta_1 L_1 \sin\beta_2 L_1}{(\beta_2^3-3\beta_1^2\beta_2)\sinh\beta_1 L_1 \sin\beta_2 L_1 + (\beta_1^3-3\beta_1\beta_2^2)\cosh\beta_1 L_1 \cos\beta_2 L_1}. \tag{46d}$$

For a given value of $M$ larger than $M^*$, eqn (45f) may be solved numerically to determine the smallest positive value of $L_1$ for which the contact force $\bar{q}_1$ remains positive in region II. When $M$ reaches the critical values $M^{**}$, then the contact force $\bar{q}_1$ vanishes at the center $z = 0$, the beam loses contact there and we must include region III in the analysis.

*Solution—contact with two free regions.* The solution is characterized by eqns (33) in region I, eqns (37)–(39) in region II, and eqns (40) in region III. For this problem matching conditions (26b)–(26d), (26f) and (26h) at $z = L_2$ yield

$$b_1 = B_1 b_3, \qquad b_2 = B_2 b_3, \qquad b_4 = B_4 b_3 \tag{47a–c}$$

$$b_5 = 0, \qquad c_2 = (B_4\beta_1+\beta_2)b_3 \tag{47d,e}$$

where constants $B_1$, $B_2$ and $B_4$ are given by

$$B_1 = (B_4\beta_1+\beta_2)L_2, \qquad B_2 = \left[\frac{(\beta_2^2-\beta_1^2)}{2\beta_1\beta_2}\right]B_1 \tag{48a,b}$$

$$B_4 = \frac{(\beta_2^3-3\beta_1^2\beta_2)}{(\beta_1^3-3\beta_1\beta_2^2)}. \tag{48c}$$

Using these results, matching conditions (25b)–(25d), (25f) and (25h) at $z = L_1$ yield

$$a_1 = A_1 b_3, \qquad a_5 = -\bar{\alpha}\left[\frac{d^2\bar{\delta}_{13}(L_1)}{dz^2} - \frac{d^2\bar{\delta}_{13}(L_2)}{dz^2}\right] \tag{49a,b}$$

$$b_3 = -B_3\left(\frac{M}{\alpha_{16}}\right) \tag{49c}$$

$$\bar{M} = \left[\frac{L}{2(L-L_1)}\right]\left[\left(\frac{B_3}{L}\right)\left(B_1 \cosh \beta_1(L_1-L_2) \cos \beta_2(L_1-L_2)\right.\right.$$
$$+B_2 \sinh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)+\cosh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)$$
$$\left.\left.+B_4 \sinh \beta_1(L_1-L_2) \cos \beta_2(L_1-L_2)-A_1\left\{\frac{2(L-L_1)^2}{3}+\frac{2\alpha_{16}}{\alpha_6}\right\}\right)+\frac{(L-L_1)}{2L}\right]^{-1} \tag{49d}$$

$$[(\beta_1^3-3\beta_1\beta_2^2)+B_4(\beta_2^3-3\beta_1^2\beta_2)] \sinh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)+[B_1(\beta_2^3-3\beta_1^2\beta_2)$$
$$+B_2(\beta_1^3-3\beta_1\beta_2^2)] \cosh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)+[B_1(\beta_1^3-3\beta_1\beta_2^2)$$
$$-B_2(\beta_2^3-3\beta_1^2\beta_2)] \sinh \beta_1(L_1-L_2) \cos \beta_2(L_1-L_2) = 0 \tag{49e}$$

where $\bar{M}$ is given by eqn (46a) and constants $A_1$ and $B_3$ are given by

$$A_1 = (\tfrac{1}{2})[B_1(\beta_1^2-\beta_2^2)+2B_2\beta_1\beta_2] \cosh \beta_1(L_1-L_2) \cos \beta_2(L_1-L_2)$$
$$+(\tfrac{1}{2})[B_2(\beta_1^2-\beta_2^2)-2B_1\beta_1\beta_2] \sinh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)$$
$$+(\tfrac{1}{2})[(\beta_1^2-\beta_2^2)-2B_4\beta_1\beta_2] \cosh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)$$
$$+(\tfrac{1}{2})[B_4(\beta_1^2-\beta_2^2)+2\beta_1\beta_2] \sinh \beta_1(L_1-L_2) \cos \beta_2(L_1-L_2) \tag{50a}$$

$$B_3 = [2A_1(L-L_1)+(B_4\beta_1+\beta_2) \cosh \beta_1(L_1-L_2) \cos \beta_2(L_1-L_2)$$
$$+(\beta_1-B_4\beta_2) \sinh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)$$
$$+(B_2\beta_1-B_1\beta_2) \cosh \beta_1(L_1-L_2) \sin \beta_2(L_1-L_2)$$
$$+(B_1\beta_1+B_2\beta_2) \sinh \beta_1(L_1-L_2) \cos \beta_2(L_1-L_2)]^{-1}. \tag{50b}$$

For a given value of $M$ larger than $M^{**}$, we guess values for $L_1$ and $L_2$ and iterate until eqns (49d) and (49e) are satisfied with $\bar{q}_1$ being nonnegative in region II. Within the context of the present solution the value of $M^{**}$ can be evaluated by specifying $L_2 = 0$ and using eqns (47), (48) and (49e) to deduce that

$$L_2 = 0, \qquad b_1 = b_2 = 0, \qquad B_1 = B_2 = 0 \tag{51a–c}$$

$$\sinh \beta_1 L_1 \sin \beta_2 L_1 = 0. \tag{51d}$$

The smallest non-trivial value of $L_1$ which satisfies eqn (51d) is

$$L_1 = \frac{\pi}{\beta_2}. \tag{52}$$

It then follows from eqns (46a), (49d), (50) and (51) that the value of $M^{**}$ becomes

$$M^{**} = M^*\left[\frac{L}{2(L-L_1)}\right]\left[\frac{(L-L_1)}{2L}-\left(\frac{B_3}{L}\right)\left(B_4 \sinh \beta_1 L_1+A_1\left\{\frac{2(L-L_1)^2}{3}+\frac{2\alpha_{16}}{\alpha_6}\right\}\right)\right]^{-1} \tag{53}$$

where constants $A_1$ and $B_3$ are given by

$$A_1 = -(\tfrac{1}{2})[B_4(\beta_1^2-\beta_2^2)+2\beta_1\beta_2]\sinh\beta_1 L_1 \tag{54a}$$

$$B_3 = [2A_1(L-L_1)-(B_4\beta_1+\beta_2)\cosh\beta_1 L_1]^{-1}, \tag{54b}$$

and where $B_4$ is given by eqn (48c).

## 4. DISCUSSION

In this section we discuss the predicted numerical results for a beam the geometry and Poisson's ratio of which are specified by

$$\frac{h}{L} = 0.1, \qquad \frac{w}{L} = 0.1, \qquad \nu = 0.25. \tag{55a–c}$$

With the use of eqns (55), the solutions of general theory (G), the Bernoulli–Euler theory (BE), the normal extensional theory (N), and the Timoshenko beam theory (T) were obtained and the results are presented graphically in Figs 2 and 3. No values are specified for the length $L$, the height $\varepsilon L$ or Young's modulus $E$ because all lengths are normalized with respect to $L$ and the moment is normalized with respect to the value $M^*$ (the same value predicted by all theories). Figure 2(a) represents a plot of the normalized length $\bar{L}_1 = L_1/L$ as a function of the normalized moment $\bar{M}$ for the initial stages of contact. In particular, notice that theory N exhibits the correct physics at the initial stages of contact because it coincides with general theory G near $\bar{M} = 1$. Also, notice from Fig. 2(a) that theory T predicts that contact begins at $\bar{M} = 1$ but has the incorrect slope there. Furthermore, theory BE predicts discontinuous behavior with no solution existing for $\bar{M}$ between the values $\bar{M} = 1$ at which contact initiates and $\bar{M} = 3$ when the contact region begins to spread. Figure 2(b) represents a plot of the normalized lengths $\bar{L}_1$ and $\bar{L}_2 = L_2/L$ for a larger range of normalized moment $\bar{M}$. Again, we observe from Fig. 2(b) that theory N exhibits the correct physics by predicting a non-zero value of $\bar{L}_2$ whereas theories T and BE predict a zero value for $\bar{L}_2$. Furthermore, we observe that for values of $\bar{M}$ larger than about three, the values of $\bar{L}_1$ predicted by theories N and T are close to those predicted by general theory G but the predictions by theory BE have considerable error.

Figure 3 shows the distribution of the contact force $\bar{q}_1$ (normalized by $M/L^2$) for three values of moment $\bar{M}$. The corresponding result for theory BE is not included in this figure because it cannot predict the contact force. For $\bar{M} = 3$ all three theories (G, N, T) predict a single free region; for $\bar{M} = 5$ both theories G and T predict a single free region, whereas theory N predicts two free regions, and for $\bar{M} = 10$ both theories G and N predict two free regions, whereas theory T predicts a single contact region. For all cases theory T significantly overestimates the maximum value of the contact force and incorrectly predicts that this maximum occurs at the edge of the contact region where physically the contact force should vanish. For the smallest value of $\bar{M}$ in Fig. 3, neither theory N which includes normal extension nor theory T which includes the effect of transverse shear deformation accurately predicts the distribution of the contact force. However, for the larger values of $\bar{M}$ it becomes clear that normal extensional theory N is superior to theory T. Moreover, for $\bar{M} = 10$, the effect of normal extension is significant near the outer edge ($z = L_1$) of the contact region, while the effect of transverse shear deformation is significant near the inner edge of the contact region ($z = L_2$).

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